The principal ideal theorem

In this section, we assume all rings are Noetherian.

Question: If $I = (a_1, \dots, a_n)$, how does $\operatorname{codim} I$ compare to n? Even more basic: What's $\operatorname{codim}(a)$? We can easily find the codimension of primes contained in principal ideals:

Claim: Any prime P properly contained in a proper principal ideal (x) has codimension 0.

Pf: Suppose $Q \subsetneq P \subsetneq (x)$, with Q prime. Then R/Q is a domain, so WLOG, assume $Q = 0$ and R is a domain.

If $y \in P$, then $y = ax$, some a, but $x \notin P$, so $a \in P$.

Thus, $P = xP \implies (1-b)P = 0$ for some $b \in (x)$. R is a domain, so $b = 1$, a contradiction. $\square$

Krull's principal ideal theorem extends this to primes minimal over principal ideals:

Krull's principal ideal Theorem: If $x \in R$ and P is minimal among primes containing x, then $\operatorname{codim} P \leq 1$.

This proof is a little more subtle, and requires a bit of background. First we state a corollary about primes minimal over a given ideal that can be concluded from results we proved about Artinian rings:

Corollary: Let R be Noetherian and $I \subseteq R$ an ideal. Let P be a prime containing I. The following are equivalent:

a.) P is minimal among primes containing I.

b) R_P/I_P is Artinian.

c.) $P_P^n \subseteq I_P$ in R_P for all $n >> 0$.

Pf: See Eisenbud Cor 2.19 — follows from theorems we proved at the beginning of the semester.

For the proof of the PIT, we need one more tool: symbolic powers of ideals.

Def: Let $Q \subseteq R$ be prime. The <u>nth symbolic power</u> of Q is

$$Q^{(n)} = Q^n R_Q \cap R = \{ r \in R \mid sr \in Q^n \text{ for some } s \in R - Q \}$$

Clearly, $Q^n \subseteq Q^{(n)} \subseteq Q$, and $(Q^{(n)})_Q = (Q_Q)^n$ (exercise)

In nice cases, this equality holds, but not always.

Ex: Let $R = k[x,y,z]/(xy - z^2)$ and $P = (x,z)$, which is prime in R. Then $y \notin P$, but $xy = z^2 \in P^2$. Thus, $x \in P^{(2)}$ but $x \notin P^2$.

Proof of the Principal ideal Theorem: Let $x \in R$, P minimal among primes containing x. We'll show that if $Q \subsetneq P$ is prime, then $\dim R_Q = 0$ so $\operatorname{codim} Q = 0$. This shows $\operatorname{codim} P \leq 1$.

Since $\operatorname{codim} P = \operatorname{codim} RP_P$, we can assume R is local and P is maximal.

Since P is minimal over (x), the above corollary says that $R/(x)$ is Artinian thus, the chain

$$(x) + Q \supseteq (x) + Q^{(2)} \supseteq (x) + Q^{(3)} \supseteq \dots$$

stabilizes at some point. Say $(x) + Q^{(n)} = (x) + Q^{(n+1)}$.

Then $Q^{(n)} \subset (x) + Q^{(n+1)}$, so for $f \in Q^{(n)}$, we can write

$$f = ax + g \quad \text{with} \quad g \in Q^{(n+1)}$$

$\Rightarrow$ $ax = f - g \in Q^{(n)} \subseteq Q$.

$\Rightarrow$ $axb \in Q^n$, some $b \in R - Q$. $x \notin Q$ by minimality of P,

so $xb \in R - Q \Rightarrow a \in Q^{(n)}$.

Thus, $Q^{(n)} \subseteq (x) Q^{(n)} + Q^{(n+1)}$. The reverse inclusion is obvious, so

$$Q^{(n)} = (x)\,Q^{(n)} + Q^{(n+1)}.$$

Thus, in $R/Q^{(n+1)}$, $\overline{Q^{(n)}} = \overline{(x)\,Q^{(n)}}$, so Nakayama says $\overline{Q^{(n)}} = 0$. i.e. $Q^{(n)} = Q^{(n+1)}$.

Thus, in R_Q, we have $(Q_Q)^n = (Q_Q)^{n+1}$, which, by Nakayama, says that $(Q_Q)^n = 0$.

By the corollary above, this implies R_Q is Artinian, and thus has dimension 0. □

We can now use this as the base case for an induction involving primes minimal over a finitely generated ideal:

Theorem: If $x_1, \dots, x_c \in R$ and P is minimal among primes containing $x_1, \dots, x_c$, then $\operatorname{codim} P \leq c$.

Pf: Again, $\operatorname{codim} P = \dim R_P$, so we can assume R is local w/ maximal ideal P.

By the above corollary, $P^n \subseteq (x_1, \dots, x_c)$ for $n \gg 0$.

let P_1 be a prime s.t. $P \supset P_1$ with no prime in between.

We'll show that P_1 is minimal over an ideal generated by $c-1$ elements. By induction, $\operatorname{codim} P_1 \leq c-1$, and we're done.

By minimality of P, P_1 cannot contain all the x_i. WLOG, assume $x_1 \notin P_1$. Then P is minimal over (P_1, x_1), so P is nilpotent in $R/(P_1, x_1)$, and, in particular, the x_i are.

Thus, there is some $n > 0$ s.t. for each $i \in \{2, \dots, c\}$, we can find $a_i \in R$, $y_i \in P_1$ s.t.

$$x_i^n = a_i x_1 + y_i$$

We want to show that P_1 is minimal among primes containing $(y_2, \dots, y_c)$.

Since P is nilpotent mod $(x_1, \dots, x_n)$, P must be nilpotent mod $(x_1, y_2, \dots, y_c)$. Thus, by the Principal ideal theorem, the image of P in $R/(y_2, \dots, y_c)$ has codimension at most one. Thus, the image of P_1 in $R/(y_2, \dots, y_c)$ has codim 0, and we're done. $\square$

This immeditely gives us a descending chain condition for prime ideals in a Noetherian ring:

Cor: Let P be a prime ideal in a Noetherian ring. Then any strictly decreasing chain of prime ideals

$$P \supsetneq P_1 \supsetneq P_2 \supsetneq \cdots$$

has finite length, bounded above by the number of generators of P.

In particular, since $(x_1, \dots, x_c) \subseteq k[x_1, \dots, x_n]$ has descending chain $(x_1, \dots, x_c) \supsetneq (x_2, \dots, x_c) \supsetneq \cdots \supsetneq (x_c) \supsetneq 0$, we know its codim is $\geq c$. The above gives us an upper bound, so we get:

Cor: The ideal $(x_1, \dots, x_c) \subseteq k[x_1, \dots, x_n]$ has codimension c.

This doesn't quite suffice yet to complute the dimension of the polynomial ring, though we will be able to soon.

There is a partial converse to the PIT:

Cor: If P is a prime of codimension c, then P is minimal over an ideal generated by c elements.

Pf: By induction, for $0 \leq r < c$, we can choose $x_1, \dots, x_r \in P$ to generate an ideal of codimsion r.

The base case is $r=0$. Any prime of codimension 0 is certainly minimal over 0.

Let $Q_1, \dots, Q_n$ be the minimal ideals contained in P. There are only finitely many as these are all associated primes of 0 (of which there are finitely many in a Noetherian ring).

By prime avoidance $P \neq \bigcup Q_i$, so $\exists\, x_1 \in P$ not in any Q_i

Then $P/(x_1)$ has codim $\leq c-1$: All the chains of maximal length in P must end w/ a minimal prime. So by induction, $P/(x_1)$ is minimal over an ideal generated by at most $c-1$ elements. Let $x_2, \dots, x_d$ be lifts of these elements in P.

Then P is minimal over $(x_1, \dots, x_d)$ w/ $d \leq c$. But $c = \operatorname{codim} P \leq d$, so $d = c$, as desired. $\square$

Recall that using primary decomposition we were able to show that if R is a Noetherian domain then

R is a UFD $\Longleftrightarrow$ every prime ideal minimal over a principal ideal is principal.

This, along w/ the PIT, leads to the following corollary:

Cor: Let R be a (Noetherian) domain. If every codimension 1 prime of R is principal, then R is a UFD.

Pf: If P is a prime minimal over a principal ideal, then it has codim 0 or 1. If it has codim 0, $P = 0$.

If it has codim 1, it's principal and we're done. $\square$